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Static and stability analyses of arbitrary straight-sided quadrilateral thin plates by DQM

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Abstract

A differential quadrature (DQ) methodology is employed for the static and stability analysis of irregular quadrilateral straight-sided thin plates. A four-noded super element is used to map the irregular physical domain into a square computational domain. Second order transformation schemes with relative ease and low computational effort are employed to transform the fourth order governing equations of thin plates between the domains. Within the domain, the displacements are the only degrees of freedom whereas, along the boundaries, the displacements as well as the second order derivatives of the displacements with respect to the associated normal coordinate variables in the computational domain are the two sets of degrees of freedom. The implementation procedures for different boundary conditions including free-edge boundaries are formulated. To demonstrate the accuracy, convergency and stability of the methodology, detailed studies of skewed and trapezoidal plates for different geometries under different boundary and loading conditions are made. Good agreement is achieved between the results of the present methodology and those of other DQ methodologies or other comparable numerical algorithms.

Keywords: Differential quadrature method; Stability; Static; Irregular; Skewed; Trapezoidal; Plates

1. Introduction

Wang et al. (1994) used the differential quadrature method (DQM) for the first time for buckling and free vibration analyses of thin skew plates. They employed the method for the analysis of simply supported and clamped plates. Using other numerical algorithms such as the finite element method (FEM), work on irregular plate problems is not new and goes back to the early stages of the development of these algorithms. Related to static and buckling analysis of skew plates, some of the early and related work may be found in references (Argyris, 1965; Jirousek and Leon, 1977; Kennedy and Prabhakara, 1979; Yetram, 1972).

Although the DQM has now established itself as an alternative numerical tool, still some major difficulties have to be dealt with. In this respect, a major difficulty arises from the implementation of the

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boundary conditions which has been explained in previous papers (Karami and Malekzadeh, 2002, in press). Methods such as the presentation of the boundary points as two very near neighboring points (the so-called δ-technique, Bert et al., 1988; Bert and Malik, 1996a,b); or building special weighting coefficients to stand for the boundary conditions (Wang and Bert, 1993); or taking the slope on the boundary as a degree of freedom (Chen et al., 1997; Wang and Gu, 1997; Wu and Liu, 1999, 2000, 2001); and the so-called CBCGE and SBCGE methods (Shu and Du, 1997a,b) are different treatments that have been introduced and implemented. Another alternative method to be introduced here and to be implemented in static and stability analyses of arbitrary straight-sided quadrilateral thin plates is claimed to be general and more accurate for these types of structures. In the present methodology, the weighting coefficients are not exclusive and any of the general schemes used in conventional DQM may be employed to determine the weighting coefficients. The applicability of the methodology to beam analysis and also to the analysis of the free vibration of irregular plates was demonstrated through previous studies (Karami and Malekzadeh, 2002, in press). It has been proved that, for cases where conventional DQMs have not yielded a convergence trend or erratic behavior, the new methodology yields accurate results with excellent convergency behavior. All kinds of boundary conditions, including free-edge boundary conditions, can be implemented in a mechanized form. Based on this methodology, the DQ static and stability analyses of irregular skew plates under all forms of loading and geometrical boundary conditions are to be addressed in this paper. The implementations of the boundary conditions in a systematic form are to be demonstrated.

2. Review of the differential quadrature method

In the method of DQ, the partial derivative of the field variable at a discrete point in the computational domain is approximated by a weighted linear sum of the values of the field variable along the line passing through that point which is parallel to the coordinate direction of the derivative. Therefore, according to the DQ rule for any function $\{\ \}$, one has,

$$\left\{ \begin{array}{c} \frac{\partial\{\ \}}{\partial\xi} \\ \frac{\partial\{\ \}}{\partial\eta} \end{array} \right\}\Bigg|_{(\xi_i,\eta_j)} = \left\{ \begin{array}{c} \sum_{m=1}^{N_\xi} A_{im}^{(\xi)}\{\ \}_{mj} \\ \sum_{n=1}^{N_\eta} A_{jn}^{(\eta)}\{\ \}_{in} \end{array} \right\} \tag{1}$$

$$\left\{ \begin{array}{c} \frac{\partial^2\{\ \}}{\partial\xi^2} \\ \frac{\partial^2\{\ \}}{\partial\eta^2} \\ \frac{\partial^2\{\ \}}{\partial\xi\,\partial\eta} \end{array} \right\}\Bigg|_{(\xi_i,\eta_j)} = \left\{ \begin{array}{l} \sum_{m=1}^{N_\xi} B_{im}^{(\xi)}\{\ \}_{mj} \\ \sum_{n=1}^{N_\eta} B_{jn}^{(\eta)}\{\ \}_{in} \\ \sum_{m=1}^{N_\xi}\ \sum_{n=1}^{N_\eta} A_{im}^{(\xi)}A_{jn}^{(\eta)}\{\ \}_{mn} \end{array} \right\} \tag{2}$$

where $A_{ij}^{(\xi)}$, $A_{ij}^{(\eta)}$ are the weighting coefficients corresponding to first order derivatives, and $B_{ij}^{(\xi)}$ and $B_{ij}^{(\eta)}$ are the weighting coefficients corresponding to second order derivatives in ξ and η directions. N_ξ and N_η are the numbers of grid points along the ξ and η axes, respectively. From the above approximations, one can realize that determination of the weighting functions plays an important role in DQM analysis. Among the many methods that have been used to evaluate the weighting coefficients in DQ analysis, the method developed primarily by Michelsen and Villadsen (1972), which was used by Shu and Richards (1992), is to be employed in this work as well. It has been claimed that this method is computationally more efficient and accurate. In fact Michelsen and Villadsen (1978) have shown that DQM's coefficients are the same as collocation method's ones when Lagrange interpolation functions are used as the test functions. For the grid points arrangements, the distributions according to Shu and Richards (1992) have been used. This has also proved to yield more accurate results.

3. The degrees of freedom

A natural coordinate system (ξ, η) for the computational domain is chosen, where $-1 \leqslant \xi, \eta \leqslant 1$. The displacement w, and the second derivative of the displacement with respect to a natural coordinate variable normal to the boundary, and only along the boundary, would be set as the two degrees of freedom of the problem. For example, along the boundary, $\xi = 1$, $\kappa^{\xi} = \partial^2 w/\partial \xi^2$ presents the second degree of freedom. Hence, in general, κ^n ($n = \xi$ or η) would play the role of an unknown parameter on the boundary. In order to incorporate the new degrees of freedom into the differential equations and facilitate the boundary conditions implementation, the higher order derivatives (derivative with order $\geqslant 2$) with respect to the coordinate system of the actual domain would be expressed in terms of κ^n ($n = \xi$ or η) and also the displacement w using the geometrical mapping procedure.

4. The geometrical mapping

Consider an arbitrary straight-sided quadrilateral plate shown in Fig. 1(a). The geometry of this plate can be mapped into a rectangular plate to be referred to as the computational domain. The coordinate axes of the quadrilateral plate which occupy the actual (or physical) domain are denoted by x and y; whereas those of the computational domain are denoted by ξ and η. The mapping process follows the standard procedure used widely in conventional finite element formulations; the physical domain is mapped into the computational domain according to the following transformation law

$$x = \sum_{i=1}^{N_s} x_i \psi_i(\xi, \eta), \qquad y = \sum_{i=1}^{N_s} y_i \psi_i(\xi, \eta) \tag{3}$$

where x_i and y_i are the coordinates of node i in the physical domain, and N_s is the number of nodal points. $\psi_i(\xi, \eta)$ is the shape function associated with node i:

$$\psi_i(\xi, \eta) = \tfrac{1}{4}(1 + \xi\xi_i)(1 + \eta\eta_i), \quad i = 1, \ldots, 4 \tag{4}$$

where ξ_i and η_i are the coordinates of nodal point i in the computational domain $\xi - \eta$. The transformation law in (4) would rule the relations between the geometries of the two domains. The derivatives of any function defined in one domain may be transformed into the other using the mapping or shape function rule defined. For example, the first, second, and third derivatives of any function { } in the

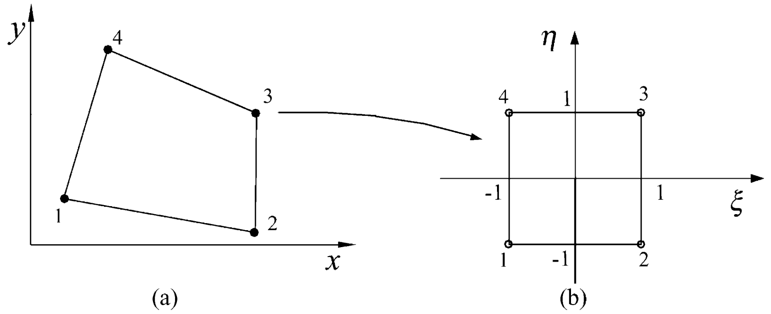

Fig. 1. (a) An arbitrary straight-sided quadrilateral plate (physical domain), (b) computational domain.

computational domain may be obtained in terms of the derivatives in the physical domain from the chain rule according to

$$\{\ \}_{,i} = \sum_{I=1}^{2} \{\ \}_{,I} x_{I,i} \tag{5}$$

$$\{\ \}_{,ij} = \sum_{I=1}^{2} \{\ \}_{,I} x_{I,ij} + \sum_{I=1}^{2} \sum_{J=1}^{2} \{\ \}_{,IJ} x_{I,i} x_{J,j} \tag{6}$$

$$\{\ \}_{,ijk} = \sum_{I=1}^{2} \{\ \}_{,I} x_{I,ijk} + \sum_{I=1}^{2} \sum_{J=1}^{2} \{\ \}_{,IJ} (x_{I,ij} x_{J,k} + x_{I,ik} x_{J,j} + x_{I,i} x_{J,jk}) + \sum_{I=1}^{2} \sum_{J=1}^{2} \sum_{K=1}^{2} \{\ \}_{,IJK} x_{I,i} x_{J,j} x_{K,k} \tag{7}$$

where $x_{I,i}$, $x_{I,ij}$ and $x_{I,ijk}$; the components of the transformation matrices for the derivatives, are related to the shape functions which map the geometry for the two coordinates. $x_{I,i}$ is a component of the so-called Jacobian transformation matrix. The inverse transformation matrices may be evaluated so that the derivatives in the physical domain may be determined in terms of the derivatives in the computational domain, so that

$$\begin{Bmatrix} \dfrac{\partial\{\ \}}{\partial x} \\ \dfrac{\partial\{\ \}}{\partial y} \end{Bmatrix} = [T_{11}] \begin{Bmatrix} \dfrac{\partial\{\ \}}{\partial \xi} \\ \dfrac{\partial\{\ \}}{\partial \eta} \end{Bmatrix} \tag{8}$$

$$\begin{Bmatrix} \dfrac{\partial^2\{\ \}}{\partial x^2} \\ \dfrac{\partial^2\{\ \}}{\partial y^2} \\ \dfrac{\partial^2\{\ \}}{\partial x\,\partial y} \end{Bmatrix} = [T_{21}] \begin{Bmatrix} \dfrac{\partial\{\ \}}{\partial \xi} \\ \dfrac{\partial\{\ \}}{\partial \eta} \end{Bmatrix} + [T_{22}] \begin{Bmatrix} \dfrac{\partial^2\{\ \}}{\partial \xi^2} \\ \dfrac{\partial^2\{\ \}}{\partial \eta^2} \\ \dfrac{\partial^2\{\ \}}{\partial \xi\,\partial \eta} \end{Bmatrix} \tag{9}$$

$$\begin{Bmatrix} \dfrac{\partial^3\{\ \}}{\partial x^3} \\ \dfrac{\partial^3\{\ \}}{\partial y^3} \\ \dfrac{\partial^3\{\ \}}{\partial x^2\,\partial y} \\ \dfrac{\partial^3\{\ \}}{\partial x\,\partial y^2} \end{Bmatrix} = [T_{31}] \begin{Bmatrix} \dfrac{\partial\{\ \}}{\partial \xi} \\ \dfrac{\partial\{\ \}}{\partial \eta} \end{Bmatrix} + [T_{32}] \begin{Bmatrix} \dfrac{\partial^2\{\ \}}{\partial \xi^2} \\ \dfrac{\partial^2\{\ \}}{\partial \eta^2} \\ \dfrac{\partial^2\{\ \}}{\partial \xi\,\partial \eta} \end{Bmatrix} + [T_{33}] \begin{Bmatrix} \dfrac{\partial^3\{\ \}}{\partial \xi^3} \\ \dfrac{\partial^3\{\ \}}{\partial \eta^3} \\ \dfrac{\partial^3\{\ \}}{\partial \xi^2\,\partial \eta} \\ \dfrac{\partial^3\{\ \}}{\partial \xi\,\partial \eta^2} \end{Bmatrix} \tag{10}$$

where $[T_{ij}]$, the inverse transformation matrices, are related to the transformation matrices according to,

$$[T_{11}] = [J_{11}]^{-1}, \quad [T_{21}] = -[J_{22}]^{-1}[J_{21}][J_{11}]^{-1}, \quad [T_{22}] = [J_{22}]^{-1}$$

$$[T_{31}] = -[J_{33}]^{-1}[J_{31}][J_{11}]^{-1}, \quad [T_{32}] = -[J_{33}]^{-1}[J_{32}][J_{22}]^{-1}, \quad [T_{33}] = [J_{33}]^{-1}$$

In Appendix A, the components of the transformation matrices, $[J_{ij}]$ are given.

5. DQ analogous of plate governing equation

The general governing equation of a thin, materially and geometrically systemic, elastic plate can be written as,

$$C_1\frac{\partial^4 w}{\partial x^4}+C_2\frac{\partial^4 w}{\partial x^3\,\partial y^2}+C_3\frac{\partial^4 w}{\partial x^2\,\partial y^2}+C_4\frac{\partial^4 w}{\partial x\,\partial y^3}+C_5\frac{\partial^4 w}{\partial y^4}-N_x\frac{\partial^2 w}{\partial x^2}-N_y\frac{\partial^2 w}{\partial y^2}-2N_{xy}\frac{\partial^2 w}{\partial x\,\partial y}+kw+\rho h\frac{\partial^2 w}{\partial t^2}$$
$$=p(x,y,t) \tag{11}$$

where, w, N_x, N_y, N_{xy}, and $p(x,y,t)$ are the transverse displacement, in plane normal and shear edge forces in x and y directions, and the intensity of transverse distributed loads, respectively. Also, C_i, ρ, h, k are, respectively, the flexural rigidity coefficients, density, thickness, and elastic stiffness of the support.

Two second order transformation processes transform the governing equation from the physical domain into the computational domain. Bert and Malik (1996a,b) used the first order transformation rule four times to do the job. The approach employed here will need less computational effort. To employ the second order transformations, K^x, K^y and K^{xy} are defined as,

$$\{K\}^{\mathrm{T}}=\{K^x\quad K^y\quad K^{xy}\}=\left\{\frac{\partial^2 w}{\partial x^2}\quad\frac{\partial^2 w}{\partial y^2}\quad\frac{\partial^2 w}{\partial x\,\partial y}\right\} \tag{12}$$

Using these definitions, the governing equation, i.e., Eq. (11), takes the following form,

$$\{C_x\}^{\mathrm{T}}\begin{Bmatrix}\dfrac{\partial^2 K^x}{\partial x^2}\\\dfrac{\partial^2 K^x}{\partial y^2}\\\dfrac{\partial^2 K^x}{\partial x\,\partial y}\end{Bmatrix}+\{C_y\}^{\mathrm{T}}\begin{Bmatrix}\dfrac{\partial^2 K^y}{\partial x^2}\\\dfrac{\partial^2 K^y}{\partial y^2}\\\dfrac{\partial^2 K^y}{\partial x\,\partial y}\end{Bmatrix}-\{N\}^{\mathrm{T}}\{K\}+kw+\rho h\frac{\partial^2 w}{\partial t^2}=p(x,y,t) \tag{13}$$

where,

$$\{N\}^{\mathrm{T}}=N_x\{1\quad L_y\quad L_{xy}\}^{\mathrm{T}}\quad\text{with}\quad L_y=N_y/N_x,\quad L_{xy}=2N_{xy}/N_x$$
$$\{C_x\}^{\mathrm{T}}=\left[C_1\quad\tfrac{1}{2}C_4\quad C_3\right]^{\mathrm{T}},\qquad\{C_y\}^{\mathrm{T}}=\left[\tfrac{1}{2}C_4\quad C_2\quad C_5\right]^{\mathrm{T}}$$

In Eq. (13), N_x is chosen as the load parameter. Other nonzero components of the plane forces may be employed as loading parameters to study the stability analysis of the equilibrium state if required. Employing the second order transformation law given by Eq. (9), the governing equation (13) becomes,

$$\{C_x\}^{\mathrm{T}}\left([T_{22}]\begin{Bmatrix}\dfrac{\partial^2 K^x}{\partial\xi^2}\\\dfrac{\partial^2 K^x}{\partial\eta^2}\\\dfrac{\partial^2 K^x}{\partial\xi\,\partial\eta}\end{Bmatrix}+[T_{21}]\begin{Bmatrix}\dfrac{\partial K^x}{\partial\xi}\\\dfrac{\partial K^x}{\partial\eta}\end{Bmatrix}\right)+\{C_y\}^{\mathrm{T}}\left([T_{22}]\begin{Bmatrix}\dfrac{\partial^2 K^y}{\partial\xi^2}\\\dfrac{\partial^2 K^y}{\partial\eta^2}\\\dfrac{\partial^2 K^y}{\partial\xi\,\partial\eta}\end{Bmatrix}+[T_{21}]\begin{Bmatrix}\dfrac{\partial K^x}{\partial\xi}\\\dfrac{\partial K^x}{\partial\eta}\end{Bmatrix}\right)$$
$$-\{N\}^{\mathrm{T}}\left([T_{21}]\begin{Bmatrix}\dfrac{\partial w}{\partial\xi}\\\dfrac{\partial w}{\partial\eta}\end{Bmatrix}+[T_{22}]\{\kappa\}\right)+kw+\rho h\frac{\partial^2 w}{\partial t^2}=p(x,y,t) \tag{14}$$

In the above equation,

$$\{\kappa\}^{\mathrm{T}} = \left\{ \frac{\partial^2 w}{\partial \xi^2} \quad \frac{\partial^2 w}{\partial \eta^2} \quad \frac{\partial^2 w}{\partial \xi \partial \eta} \right\}^{\mathrm{T}}$$

For grid points within the domain, the second order derivatives are expressed in terms of the displacement, w, that is,

$$\{\kappa\}|_{(\xi_i,\eta_j)} = \left\{ \begin{array}{l} \sum_{m=1}^{N_\xi} B_{im}^{(\xi)} w_{mj} \\ \sum_{n=1}^{N_\eta} B_{jn}^{(\eta)} w_{in} \\ \sum_{m=1}^{N_\xi} \sum_{n=1}^{N_\eta} A_{im}^{(\xi)} A_{jn}^{(\eta)} w_{mn} \end{array} \right\} \quad \text{for } i = 2,3,\ldots,N_\xi - 1, \text{ and } j = 2,3,\ldots,N_\eta - 1 \tag{15}$$

Using Eqs. (1), (2), and (15), the DQ analogue to the plate governing equation can be written:

$$\sum_{m=1}^{N_\xi} \sum_{n=1}^{N_\eta} \left[H_{mn}^{(x)} \quad H_{mn}^{(y)} \right] \begin{bmatrix} K_{mn}^x \\ K_{mn}^y \end{bmatrix} - N_x \sum_{m=1}^{N_\xi} \sum_{n=1}^{N_\eta} R_{mn} w_{mn} + k w_{ij} + \rho h \frac{\partial^2 w_{ij}}{\partial t^2} = p(x_i, y_j, t) = p_{ij}$$
$$\text{for } i = 2,3,\ldots,N_\xi - 1, \text{ and } j = 2,3,\ldots,N_\eta - 1 \tag{16}$$

where

$$H_{mn}^{(x)} = \{C_x\}^{\mathrm{T}} \left([T_{21}]_{ij} \left\{ \begin{array}{c} A_{im}^{(\xi)} \delta_{jn} \\ A_{jn}^{(\eta)} \delta_{im} \end{array} \right\} + [T_{22}]_{ij} \left\{ \begin{array}{c} B_{im}^{(\xi)} \delta_{jn} \\ B_{jn}^{(\eta)} \delta_{im} \\ A_{im}^{(\xi)} A_{jn}^{(\eta)} \end{array} \right\} \right)$$

$$H_{mn}^{(y)} = \{C_y\}^{\mathrm{T}} \left([T_{21}]_{ij} \left\{ \begin{array}{c} A_{im}^{(\xi)} \delta_{jn} \\ A_{jn}^{(\eta)} \delta_{im} \end{array} \right\} + [T_{22}]_{ij} \left\{ \begin{array}{c} B_{im}^{(\xi)} \delta_{jn} \\ B_{jn}^{(\eta)} \delta_{im} \\ A_{im}^{(\xi)} A_{jn}^{(\eta)} \end{array} \right\} \right)$$

$$R_{mn} = \{L\}^{\mathrm{T}} \left([T_{21}]_{ij} \left\{ \begin{array}{c} A_{im}^{(\xi)} \delta_{jn} \\ A_{jn}^{(\eta)} \delta_{im} \end{array} \right\} + [T_{22}]_{ij} \left\{ \begin{array}{c} B_{im}^{(\xi)} \delta_{jn} \\ B_{jn}^{(\eta)} \delta_{im} \\ A_{im}^{(\xi)} A_{jn}^{(\eta)} \end{array} \right\} \right)$$
$$\text{for } i = 2,3,\ldots,N_\xi - 1, \text{ and } j = 2,3,\ldots,N_\eta - 1$$

A second order transformation will be used to transform K^x and K^y from the physical domain into the computational domain. To do so, one may employ Eq. (9) in the following form at any arbitrary grid point (ξ_m, η_n):

$$\left\{ \begin{array}{c} K^x \\ K^y \end{array} \right\}_{mn} = [\overline{T}_{21}]_{mn} \left\{ \begin{array}{c} \dfrac{\partial w}{\partial \xi} \\ \dfrac{\partial w}{\partial \eta} \end{array} \right\}_{mn} + [\overline{T}_{22}]_{mn} \left\{ \begin{array}{c} \dfrac{\partial^2 w}{\partial \xi^2} \\ \dfrac{\partial^2 w}{\partial \eta^2} \\ \dfrac{\partial^2 w}{\partial \xi \partial \eta} \end{array} \right\}_{mn} \tag{17}$$

where $[\overline{T}_{21}]$ and $[\overline{T}_{22}]$ are the reduced form of the second order transformation $[T_{21}]$ and $[T_{22}]$. Employing the above relation, Eq. (16) may be written as

$$\sum_{m=1}^{N_\xi}\sum_{n=1}^{N_\eta}\left[H_{mn}^{(x)} \quad H_{mn}^{(y)}\right][\overline{T}_{21}]_{mn}\left\{\begin{array}{c}\dfrac{\partial w}{\partial \xi}\\ \dfrac{\partial w}{\partial \eta}\end{array}\right\}_{mn} + \sum_{m=1}^{N_\xi}\sum_{n=1}^{N_\eta}\left[H_{mn}^{(x)} \quad H_{mn}^{(y)}\right][\overline{T}_{22}]_{mn}\left\{\begin{array}{c}\dfrac{\partial^2 w}{\partial \xi^2}\\ \dfrac{\partial^2 w}{\partial \eta^2}\\ \dfrac{\partial^2 w}{\partial \xi\,\partial \eta}\end{array}\right\}_{mn}$$

$$- N_x\sum_{m=1}^{N_\xi}\sum_{n=1}^{N_\eta} R_{mn}w_{mn} + kw_{ij} + \rho h\frac{\partial^2 w_{ij}}{\partial t^2} = p_{ij} \quad \text{for } i = 2,3,\ldots,N_\xi - 1, \text{ and } j = 2,3,\ldots,N_\eta - 1 \tag{18}$$

Using quadrature rule for the first and second orders derivatives (except for those which are chosen as the degrees of freedom at the boundary points), one may reduce the governing equation to a standard form.

$$[S_{\mathrm{db}}]\{U_{\mathrm{b}}\} + [S_{\mathrm{dd}}]\{U_{\mathrm{d}}\} + N_x([\overline{B}_{\mathrm{db}}]\{\overline{U}_{\mathrm{b}}\} + [\overline{B}_{\mathrm{dd}}]\{U_{\mathrm{d}}\}) + \rho\left\{\frac{\partial^2 U_{\mathrm{d}}}{\partial t^2}\right\} = \{p\} \tag{19}$$

In the above equation,

$$\{U_{\mathrm{b}}\} = \left\{\begin{array}{c}\{w\}_{\mathrm{b}}\\ \{\kappa\}_{\mathrm{b}}\end{array}\right\}, \qquad \{U_{\mathrm{d}}\} = \{w\}_{\mathrm{d}}, \qquad \{\overline{U}_{\mathrm{b}}\} = \{w\}_{\mathrm{b}}$$

The subscript b denotes a boundary point whereas d represents a domain grid point.

6. Implementation of boundary conditions

In the following section, the DQ analogues of three types of boundary conditions, i.e., simply supported (S), clamped (C), and free edges (F) will be presented. In order to simplify the notations in the DQ analogues, we use the indices b_ξ for the edges $\xi = \pm 1$ which take the value of 1 at the edge $\xi = -1$ and N_ξ at the edge $\xi = 1$, respectively. Similarly, b_η will be used for the edges $\eta = \pm 1$, in which $b_\eta = 1$ at $\eta = -1$ and $b_\eta = N_\eta$ at the edge $\eta = 1$, respectively.

6.1. Simply supported

For simply supported edges, the boundary conditions are:

$$w = 0, \quad M_n = 0 \tag{20}$$

The bending moment M_n can be expressed in terms of the Cartesian components of the moments at that point as (Timoshenko and Woinowsky-Krieger, 1970);

$$M_n = n_x^2 M_x + n_y^2 M_y + 2n_x n_y M_{xy}$$

where n_x and n_y are, respectively the x and y components of the unit normal to the boundary. Eq. (20) may be formatted as,

$$M_n = \{\overline{n}\}^{\mathrm{T}}\{M\} = \{\overline{n}\}^{\mathrm{T}}[\overline{D}]\{K\} \tag{21}$$

where,

$$\{\overline{n}\}^{\mathrm{T}} = \{\, n_x^2 \quad n_y^2 \quad 2n_xn_y \,\}, \qquad [\overline{D}] = [\overline{I}][D], \quad [\overline{I}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$[D]$ is the bending stiffness matrix of the plate (see Appendix B). Using the second order transformation law from Eq. (9), Eq. (21) reads,

$$\{\overline{M}_1\}^{\mathrm{T}} \begin{Bmatrix} \dfrac{\partial w}{\partial \xi} \\ \dfrac{\partial w}{\partial \eta} \end{Bmatrix} + \{\overline{M}_2\}^{\mathrm{T}} \begin{Bmatrix} \dfrac{\partial^2 w}{\partial \xi^2} \\ \dfrac{\partial^2 w}{\partial \eta^2} \\ \dfrac{\partial^2 w}{\partial \xi \partial \eta} \end{Bmatrix} = 0 \tag{22}$$

where

$$\{\overline{M}_1\}^{\mathrm{T}} = \{\overline{n}\}^{\mathrm{T}}[\overline{D}][T_{21}], \qquad \{\overline{M}_2\}^{\mathrm{T}} = \{\overline{n}\}^{\mathrm{T}}[\overline{D}][T_{22}]$$

If edges $\xi = -1$ or $\xi = 1$ are simply supported, the DQ analogues of the first equation in (20) will take the form

$$w_{b_\xi J} = 0 \quad \text{for } J = 2, 3, \ldots, N_\xi - 1 \ \text{ and } \ b_\xi = 1 \ \text{ or } \ b_\xi = N_\xi \tag{23}$$

whereas the DQ analogues of the second equation (20) will be

$$\{\overline{M}_1\}^{\mathrm{T}}_{b_\xi J} \begin{Bmatrix} \sum_{m=1}^{N_\xi} A^{(\xi)}_{b_\xi m} w_{mJ} \\ \sum_{n=1}^{N_\eta} A^{(\eta)}_{Jn} w_{b_\xi n} \end{Bmatrix} + \{\overline{M}_2\}^{\mathrm{T}}_{b_\xi J} \begin{Bmatrix} \kappa^{\xi}_{b_\xi J} \\ \sum_{n=1}^{N_\eta} B^{(\eta)}_{Jn} w_{b_\xi n} \\ \sum_{m=1}^{N_\xi} \sum_{n=1}^{N_\eta} A^{(\xi)}_{b_\xi m} A^{(\eta)}_{Jn} w_{mn} \end{Bmatrix} = 0 \tag{24}$$

In the same way for edges $\eta = -1$ or $\eta = 1$:

$$w_{Ib_\eta} = 0 \quad \text{for } I = 2, 3, \ldots, N_\eta - 1 \ \text{ and } \ b_\eta = 1 \ \text{ or } \ b_\eta = N_\eta \tag{25}$$

$$\{\overline{M}_1\}^{\mathrm{T}}_{Ib_\eta} \begin{Bmatrix} \sum_{m=1}^{N_\xi} A^{(\xi)}_{Im} w_{mb_\eta} \\ \sum_{n=1}^{N_\eta} A^{(\eta)}_{b_\eta n} w_{In} \end{Bmatrix} + \{\overline{M}_2\}^{\mathrm{T}}_{Ib_\eta} \begin{Bmatrix} \sum_{n=1}^{N_\eta} B^{(\eta)}_{Im} w_{mb_\eta} \\ \kappa^{\eta}_{Ib_\eta} \\ \sum_{m=1}^{N_\xi} \sum_{n=1}^{N_\eta} A^{(\xi)}_{Im} A^{(\eta)}_{b_\eta n} w_{mn} \end{Bmatrix} = 0 \tag{26}$$

6.2. Clamped

The boundary conditions for clamped edges can be stated as

$$w = 0, \quad \frac{\partial w}{\partial n} = 0 \tag{27}$$

where n stands for ξ and η.

If the edges $\xi = -1$ or $\xi = 1$ will be clamped, the DQ analogues of the first equation in (27) become

$$w_{b_\xi J} = 0 \quad \text{for } J = 2, 3, \ldots, N_\xi - 1 \ \text{ and } \ b_\xi = 1 \ \text{ or } \ b_\xi = N_\xi \tag{28}$$

A zero slope boundary condition would be implemented through κ^ξ. For example, along the edge $\xi = -1$ the condition $\partial w/\partial \xi = 0$ might be implemented by

$$\kappa^{\xi}_{b_\xi J} - \sum_{M=\mathrm{ml}}^{\mathrm{mu}} A^{(\xi)}_{b_\xi M} \frac{\partial w}{\partial \xi}\bigg|_{(\xi_M,\eta_J)} = 0 \quad \text{for } J = 2,3,\ldots,N_\xi - 1 \tag{29}$$

where ml $= 2$ for a zero slope condition at $\xi = -1$, otherwise ml $= 1$. For a zero slope condition at edge $\xi = 1$, mu $= N_\xi - 1$, otherwise, mu $= N_\xi$. Also, as stated before, $b_\xi = 1$ for $\xi = -1$ and $b_\xi = N_\xi$ for $\xi = 1$. The slope term in the summation in Eq. (29) can be expanded

$$\kappa^{\xi}_{b_\xi J} - \sum_{M=\mathrm{ml}}^{\mathrm{mu}} \sum_{n=1}^{N_\xi} A^{(\xi)}_{b_\xi M} A^{(\xi)}_{Mn} w_{nJ} = 0 \quad \text{for } J = 2,3,\ldots,N_\xi - 1 \tag{30}$$

In a similar way, one can implement the clamped boundary condition at edge $\eta = -1$ or $\eta = 1$; the results read

$$w_{Ib_\eta} = 0 \quad \text{for } I = 2,3,\ldots,N_\eta - 1 \ \text{ and } \ b_\eta = 1 \text{ or } b_\eta = N_\eta \tag{31}$$

$$\kappa^{\eta}_{Ib_\eta} - \sum_{M=\mathrm{ml}}^{\mathrm{mu}} \sum_{n=1}^{N_\eta} A^{(\eta)}_{b_\eta M} A^{(\eta)}_{Mn} w_{In} = 0 \quad \text{for } J = 2,3,\ldots,N_\eta - 1 \tag{32}$$

where ml $= 2$ if a zero slope condition applies to the edge $\eta = -1$, otherwise, ml $= 1$. For a zero slope condition at edge $\eta = 1$, ml $= N_\eta - 1$, otherwise ml $= N_\eta$. Also, $b_\eta = 1$ for $\eta = -1$ and $b_\eta = N_\eta$ for $\eta = 1$.

6.3. Free-edge boundary condition

For a free boundary condition the conditions (Timoshenko and Woinowsky-Krieger, 1970)

$$M_n = 0, \quad Q_n + \frac{\partial M_{sn}}{\partial s} = 0 \tag{33}$$

should be satisfied. s and n are the coordinate variables of the axes tangent and normal to the boundary, Q_n is shear force and M_{sn} is the twisting moment at the edges (Timoshenko and Woinowsky-Krieger, 1970). The DQ analogue of M_n is given by Eqs. (24) and (26). The DQ analogues of the effective shear stresses will be derived based on a new approach (Karami and Malekzadeh, in press). Eq. (33) may be written as,

$$\{F\}^{\mathrm{T}} \left\{ \begin{array}{c} \dfrac{\partial^3 w}{\partial x^3} \\ \dfrac{\partial^3 w}{\partial y^3} \\ \dfrac{\partial^3 w}{\partial x^2 \partial y} \\ \dfrac{\partial^3 w}{\partial x \partial y^2} \end{array} \right\} = 0 \tag{34}$$

The details of derivation and also the definition of matrix $\{F\}$ are given in Appendix B. Using Eq. (10), Eq. (34) becomes

$$\{\overline{F}_3\}^{\mathrm{T}} \left\{ \begin{array}{c} \dfrac{\partial^3 w}{\partial \xi^3} \\ \dfrac{\partial^3 w}{\partial \eta^3} \\ \dfrac{\partial^3 w}{\partial \xi^2 \partial \eta} \\ \dfrac{\partial^3 w}{\partial \xi \partial \eta^2} \end{array} \right\} + \{\overline{F}_2\}^{\mathrm{T}} \left\{ \begin{array}{c} \dfrac{\partial^2 w}{\partial \xi^2} \\ \dfrac{\partial^2 w}{\partial \eta^2} \\ \dfrac{\partial^2 w}{\partial \xi \partial \eta} \end{array} \right\} + \{\overline{F}_1\}^{\mathrm{T}} \left\{ \begin{array}{c} \dfrac{\partial w}{\partial \xi} \\ \dfrac{\partial w}{\partial \eta} \end{array} \right\} = 0 \tag{35}$$

where

$$\{\overline{F}_3\} = \{F\}^{\mathrm{T}}[T_{33}], \quad \{\overline{F}_2\} = \{F\}^{\mathrm{T}}[T_{32}], \quad \{\overline{F}_1\} = \{F\}^{\mathrm{T}}[T_{31}]$$

Along the edges $\xi = -1$ or $\xi = 1$, the DQ rule can be applied to Eq. (35) to derive DQ analogues of zero effective shear forces,

$$\{\overline{F}_3\}^{\mathrm{T}}_{b_\xi J}\left\{\begin{array}{c} \sum_{m=1}^{N_\xi} C^{(\xi)}_{b_\xi m} w_{mJ} \\ \sum_{n=1}^{N_\xi} C^{(\xi)}_{jn} w_{b_\xi n} \\ \sum_{m=1}^{N_\xi}\sum_{n=1}^{N_\eta} B^{(\xi)}_{b_\xi m} A^{(\xi)}_{Jn} w_{mn} \\ \sum_{m=1}^{N_\xi}\sum_{n=1}^{N_\eta} A^{(\xi)}_{b_\xi m} B^{(\xi)}_{Jn} w_{mn} \end{array}\right\} + \{\overline{F}_2\}^{\mathrm{T}}_{b_\xi J}\left\{\begin{array}{c} \kappa^{\xi}_{b_\xi J} \\ \sum_{n=1}^{N_\xi} B^{(\xi)}_{jn} w_{b_\xi n} \\ \sum_{m=1}^{N_\xi}\sum_{n=1}^{N_\eta} A^{(\xi)}_{b_\xi m} A^{(\xi)}_{Jn} w_{mn} \end{array}\right\}$$
$$+ \{\overline{F}_1\}^{\mathrm{T}}_{b_\xi J}\left\{\begin{array}{c} \sum_{m=1}^{N_\xi} A^{(\xi)}_{b_\xi m} w_{mJ} \\ \sum_{n=1}^{N_\xi} A^{(\xi)}_{b_\xi n} w_{b_\xi n} \end{array}\right\} = 0 \quad \text{for } J = 2, 3, \ldots, N_\xi - 1 \tag{36}$$

The above relations along the edges $\eta = -1$ or $\eta = 1$ can be written similarly:

$$\{\overline{F}_3\}^{\mathrm{T}}_{Ib_\eta}\left\{\begin{array}{c} \sum_{m=1}^{N_\xi} C^{(\eta)}_{Im} w_{mb_\eta} \\ \sum_{n=1}^{N_\eta} C^{(\eta)}_{b_\eta n} w_{In} \\ \sum_{m=1}^{N_\xi}\sum_{n=1}^{N_\eta} B^{(\eta)}_{Im} A^{(\eta)}_{b_\eta n} w_{mn} \\ \sum_{m=1}^{N_\xi}\sum_{n=1}^{N_\eta} A^{(\eta)}_{Im} B^{(\eta)}_{b_\eta n} w_{mn} \end{array}\right\} + \{\overline{F}_2\}^{\mathrm{T}}_{Ib_\eta}\left\{\begin{array}{c} \sum_{n=1}^{N_\xi} B^{(\eta)}_{Im} w_{mb_\eta} \\ \kappa^{\eta}_{Ib_\eta} \\ \sum_{m=1}^{N_\xi}\sum_{n=1}^{N_\eta} A^{(\eta)}_{Im} A^{(\eta)}_{b_\eta n} w_{mn} \end{array}\right\}$$
$$+ \{\overline{F}_1\}^{\mathrm{T}}_{Ib_\eta}\left\{\begin{array}{c} \sum_{m=1}^{N_\xi} A^{(\eta)}_{Im} w_{mb_\eta} \\ \sum_{n=1}^{N_\eta} A^{(\eta)}_{b_\eta n} w_{In} \end{array}\right\} = 0 \quad \text{for } I = 2, 3, \ldots, N_\eta - 1 \tag{37}$$

7. The stability analysis

The displacements on the boundaries may be written as

$$\{w\}_{\mathrm{b}} = [\overline{S}_{\mathrm{bb}}]\{U_{\mathrm{d}}\} \tag{38}$$

where $[\overline{S}_{\mathrm{bb}}]$ is obtained by partitioning and using matrix algebra on the coefficients of the system of equations. Using Eq. (38) and eliminating the boundary terms from the governing equation, one has

$$([\overline{S}] + N_x[\overline{B}])\{U_{\mathrm{d}}\} = \{0\} \tag{39}$$

In the above equation,

$$[\overline{S}] = [S_{\mathrm{db}}][\overline{S}_{\mathrm{bb}}] + [S_{\mathrm{dd}}], \quad [\overline{B}] = [\overline{B}_{\mathrm{db}}][\overline{S}_{\mathrm{bb}}] + [\overline{B}_{\mathrm{dd}}]$$

From Eq. (39), the critical buckling load as well as the mode shapes may be obtained.

8. The static analysis

The same procedure used for stability analysis may be implemented to obtain the standard form of the static analysis system of equations as

$$\begin{bmatrix} [S_{\rm bb}] & [S_{\rm bd}] \\ [S_{\rm db}] & [S_{\rm dd}] \end{bmatrix} \begin{Bmatrix} \{U_{\rm b}\} \\ \{U_{\rm d}\} \end{Bmatrix} = \begin{Bmatrix} \{0\} \\ \{p\} \end{Bmatrix} \tag{40}$$

By eliminating the boundary degrees of freedom, one has

$$[S]\{U_{\rm d}\} = \{p\} \tag{41}$$

where $[S] = [S_{\rm dd}] - [S_{\rm db}][S_{\rm bb}]^{-1}[S_{\rm bd}]$. If, on the boundaries, there exists any loading condition, then Eq. (41) may be modified to include these effects as

$$[S_{\rm bb}]\{U_{\rm b}\} + [S_{\rm bd}]\{U_{\rm d}\} = \{p_{\rm b}\} \tag{42}$$

In the above equation, $\{p_{\rm b}\}$ represents the applied loads on the boundaries. Also, Eq. (40) takes the following form

$$[S]\{U_{\rm d}\} = \{P\} \tag{43}$$

where

$$\{p\} - [S_{\rm db}][S_{\rm bb}]^{-1}\{p_{\rm b}\} = \{P\}$$

After evaluation of the displacements, moments and shear forces and various components of stress can be obtained.

9. Numerical results

In order to demonstrate the efficiency of the methodology for the static and stability analyses of irregular straight-sided quadrilateral plates, several different plate examples problems, i.e., skew plates, symmetric and unsymmetric trapezoidal plates under different boundary conditions and loading conditions, and for different skew angles, aspect ratios, and cord ratios will be studied (see Fig. 2). The Poisson's ratio is assumed to be 0.3 for all examples unless otherwise specified.

9.1. Static analysis results

In Table 1, the deflection ($\overline{w} = (Dw)/(qa^4 \sin\theta)$) and the bending moments ($\overline{M}_x = (DM_x)(qa^2 \sin\theta)$), ($\overline{M}_y = (DM_y)(qa^2 \sin\theta)$) at the midpoint of a S–C–S–C rhombic plate under a uniform loading q are shown. The results are compared with those from the spline-FEM (SFEM) given by Tham et al. (1988) and those

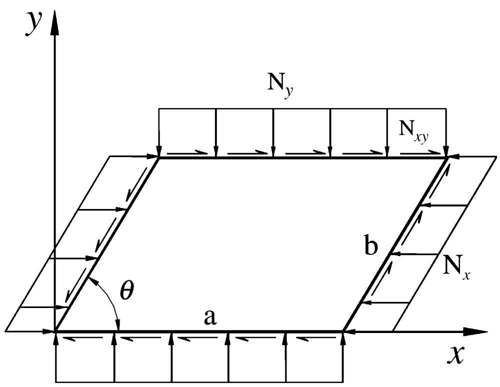

Fig. 2. Skew plate under in-plane loading.

Table 1
Deflections and bending moments at midpoint of S–C–S–C rhombic plate

Parameter	Method	N_ξ	Skew angle				
			90°	75°	60°	45°	30°
$\overline{w}$	Present	11	0.1917	0.1764	0.1355	0.0821	0.00336
		15	0.1917	0.1764	0.1355	0.0821	0.00337
		19	0.1917	0.1764	0.1355	0.0821	0.00338
	Tham et al. (1988)		0.192	0.176	0.135	0.0815	0.0327
	Wang and Hsu (1994)		0.192	0.176	0.135	0.0814	0.0326
$\overline{M}_x$	Present	11	0.02439	0.02335	0.02033	0.01561	0.00975
		15	0.02439	0.02334	0.02033	0.01563	0.00981
		19	0.02439	0.02334	0.02033	0.01565	0.00986
	Tham et al. (1988)		0.0244	0.0233	0.0203	0.0156	0.0098
	Wang and Hsu (1994)		0.0244	0.0234	0.0204	0.0157	0.0097
$\overline{M}_y$	Present	11	0.03324[a]	0.03272	0.03096	0.02743	0.0212
		15	0.03324	0.03271	0.03096	0.02746	0.0214
		19	0.03324	0.03271	0.03096	0.02748	0.0215
	Tham et al. (1988)		0.0333	0.0328	0.0310	0.0276	0.0216
	Wang and Hsu (1994)		0.0333	0.0329	0.0312	0.0278	0.0217

[a] Exact solution = 0.0332 (Timoshenko and Woinowsky-Krieger, 1970).

using B3-spline functions as given by Wang and Hsu (1994). The exact solutions for $\overline{M}_y$ (Timoshenko and Woinowsky-Krieger, 1970) do compare very well.

In Table 2, the deflection and moments again at the midpoint for a S–C–S–C supported irregular quadrilateral plate (see Fig. 3) under a uniformly distributed load ($p = 1$, $D = 1$) are shown. The results are compared with those from the B3-spline method given by Wang and Hsu (1994).

In Table 3, the convergence behavior of the method is studied by determining the static results for a fully clamped skew plate (C–C–C–C) with $a/b = 1$ which are compared with the solutions of two different forms of finite elements formulations given by Sengupta (1995) and Butalia et al. (1990).

Again, in Table 4, the convergence behavior of the method is studied for the static analysis of a skew plate with two opposite edges free and the other edges clamped, i.e., C–F–C–F with $a/b = 1$ at different skew angles. The results are compared with those given by Butalia et al. (1990) and with finite elements results given by Sengupta (1995). In all cases, one can find that the method converges to accurate solutions.

Table 2
Deflections and moments at midpoint for an irregular quadrilateral plate under uniformly distributed load ($p = 1$, $D = 1$)

Method	N_ξ	w	M_x	M_y	M_{xy}
Present	5	61.5593	6.14657	3.92910	0.17738
	7	58.9701	5.95737	3.68566	0.20785
	9	58.9423	5.95013	3.69177	0.20762
	11	58.9418	5.94970	3.69271	0.20539
	13	58.9411	5.94969	3.69269	0.20496
	15	58.9408	5.94966	3.69268	0.20494
	17	58.9407	5.94964	3.69268	0.20491
	19	58.9406	5.94964	3.69268	0.20487
	21	58.9406	5.94963	3.69270	0.20485
Wang and Hsu (1994)		58.929	5.984	3.7006	0.2070

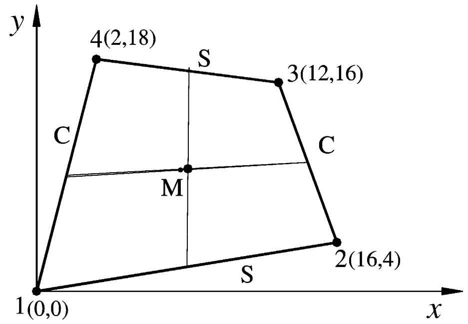

Fig. 3. The geometry of the irregular plate.

Table 3
Convergence study of static results for fully clamped skew plates ($a/b = 1$)

θ	Method	N_ξ	$((100D)/(qa^4))w$	M_1/qa^2	M_2/qa^2
75°	Present	11	0.11229	0.022796	0.020210
		15	0.11229	0.022798	0.020214
		19	0.11229	0.022798	0.020215
	FEM (Sengupta, 1995)		0.11321	0.022574	0.020064
	Elem. A (Sengupta, 1995)		0.11252	0.022814	0.020214
	Elem. B (Sengupta, 1995)		0.11606	0.022373	0.020446
			0.11217	0.023052	0.020395
60°	Present	11	0.076883	0.019756	0.015452
		15	0.076899	0.019781	0.015439
		19	0.076899	0.019774	0.015439
	FEM (Sengupta, 1995)		0.077619	0.019595	0.015300
	Elem. A (Sengupta, 1995)		0.077094	0.019790	0.015438
	Elem. B (Sengupta, 1995)		0.076762	0.019520	0.015058
	Butalia et al. (1990)		0.076756	0.019976	0.015568
45°	Present	11	0.037681	0.014390	0.009801
		15	0.037684	0.014425	0.009753
		19	0.037685	0.014435	0.009749
	FEM (Sengupta, 1995)		0.038044	0.014294	0.009624
	Elem. A (Sengupta, 1995)		0.037819	0.014439	0.009751
	Elem. B (Sengupta, 1995)		0.037700		
	Butalia et al. (1990)		0.037481	0.014567	0.009733
30°	Present	15	0.010820	0.008006	0.004541
		19	0.010821	0.008012	0.004535
		25	0.010822	0.008020	0.004534
	FEM (Sengupta, 1995)		0.010912	0.007938	0.004446
	Elem. A (Sengupta, 1995)		0.010894	0.008022	0.004551
	Elem. B (Sengupta, 1995)		0.010856	0.007930	0.004370
	Butalia et al. (1990)		0.010650	0.008151	0.004410

In Table 5, the method's convergence is tested at different numbers of grid points for the static analysis results of a C–F–C–F trapezoidal plate at different cord ratios of c/b with $\beta = 0$ and $b/a = 1$ (see Fig. 4). The converged solutions are compared with those given by Liew (1992) results.

Table 4
Convergence study of static results for skew plates with two opposite edges free and other edges clamped (C–F–C–F; $a/b = 1$)

θ	Method	N_ξ	$((100D)/(qa^4))w$	M_1/qa^2	M_2/qa^2
75°	Present	11	0.226923	0.038293	0.0094680
		15	0.227157	0.038340	0.0094663
		19	0.227216	0.038353	0.0094631
		21	0.227226	0.038357	0.0094624
	FEM (Sengupta, 1995)		0.227794	0.038026	0.0093685
	Elem. A (Sengupta, 1995)		0.227575	0.038364	0.0094657
	Elem. B (Sengupta, 1995)		0.225850	0.039029	0.009592
	Butalia et al. (1990)		0.227469	0.039410	0.009806
60°	Present	15	0.15552	0.031790	0.005480
		19	0.15564	0.031847	0.005520
		23	0.15572	0.031883	0.005538
		25	0.15574	0.031886	0.005545
	FEM (Sengupta, 1995)		0.15632	0.031694	0.005598
	Elem. A (Sengupta, 1995)		0.15612	0.031989	0.005643
	Elem. B (Sengupta, 1995)		0.15507	0.032639	0.005684
	Butalia et al. (1990)		0.15609	0.033130	0.006076
45°	Present	13	0.073987	0.021747	0.000059
		15	0.074775	0.021924	0.000065
		19	0.075595	0.022115	0.000072
		25	0.076092	0.022257	0.000078
	FEM (Sengupta, 1995)		0.076750	0.022313	0.000103
	Elem. A (Sengupta, 1995)		0.076756	0.022572	0.000102
	Elem. B (Sengupta, 1995)		0.076724	0.023023	0.000089
	Butalia et al. (1990)		0.079819	0.023807	0.000180
30°	Present	19	0.0207	0.011710	0.002756
		21	0.021010	0.011814	0.002656
		25	0.021407	0.011961	0.002501
	FEM (Sengupta, 1995)		0.022025	0.011851	0.0021587
	Elem. A (Sengupta, 1995)		0.022131	0.012067	0.0022955
	Elem. B (Sengupta, 1995)		0.021912	0.012175	0.0024535
	Butalia et al. (1990)		0.021800	0.012991	0.0007655

Table 5
Convergence and accuracy study of C–F–C–F trapezoidal plates ($\beta = 0$; $b/a = 1$)

Method	N_ξ	$c/b = 0.3$			$c/b = 0.5$		
		$((100D)/(qa^4))w$	M_1/qa^2	M_2/qa^2	$((100D)/(qa^4))w$	M_1/qa^2	M_2/qa^2
Present	11	0.2808	0.041518	0.008070	0.2662	0.040978	0.008781
	13	0.2815	0.041591	0.008009	0.2664	0.041004	0.008755
	15	0.2819	0.041608	0.008000	0.2665	0.041007	0.008746
	17	0.2820	0.041617	0.007988	0.2666	0.041010	0.008738
	19	0.2822	0.041623	0.007980	0.2666	0.041010	0.008738
	21	0.2822	0.041625	0.007975	0.2666	0.041010	0.008738
Liew (1992)		0.2816	0.041253	0.007626	0.2662	0.040743	0.008637

In Table 6, the method convergence and accuracy is again tested for unsymmetric (skew) simply supported trapezoidal plates ($a/b = 1.5$; $c/b = 0.4$) at two different values of β angle (see Fig. 4). It seems even 11 grid points in each direction would be enough for a very accurate solution.

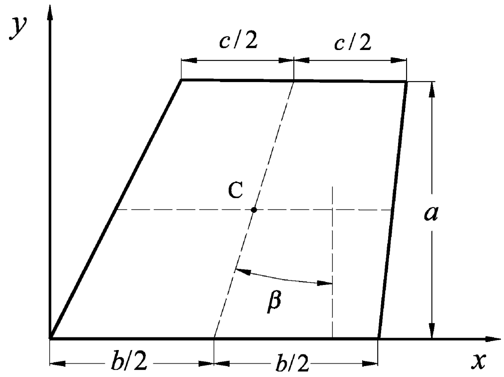

Fig. 4. Geometry of trapezoidal plate.

Table 6
Convergence study of static results for unsymmetric simply supported trapezoidal plates ($a/b = 1.5$; $c/b = 0.4$)

N_ξ	$\beta = 10°$			$\beta = 20°$		
	$((100D)/(qa^4))w$	M_1/qa^2	M_2/qa^2	$((100D)/(qa^4))w$	M_1/qa^2	M_2/qa^2
11	0.004733	0.02213	0.009401	0.004147	0.02098	0.00834
13	0.004733	0.02213	0.009401	0.004147	0.02098	0.00834
15	0.004733	0.02213	0.009402	0.004147	0.02099	0.00834
17	0.004733	0.02213	0.009402	0.004147	0.02099	0.00835
19	0.004733	0.02213	0.009402	0.004147	0.02099	0.00835

9.2. Stability analysis results

In Table 7, again convergence and accuracy for the solutions for the critical buckling load ($K = (N_xa^2)/(\pi^2D)$) of a symmetric trapezoidal plate with $a/b = 1$ under three different boundary conditions with different cord ratios are presented. Exact solutions and Levy's solution (Levy, 1942) are given for comparisons.

In Table 8, the results for the critical buckling load ($K = (N_xa^2)/(\pi^2D)$) of a symmetric trapezoidal plate but under different forms of loading condition, i.e., $N_x = N_y \neq 0$, are presented.

In Table 9, a convergence study is carried out on the critical buckling load ($K = (N_xa^2)/(\pi^2D)$) measurement for unsymmetrical trapezoidal plates at different angles for β under the loading condition $N_x = N_y \neq 0$.

In Table 10, accuracy and convergence of the critical buckling load ($K = (N_xa^2)(\pi^2D)$) for a simply supported skew plate with skew angle $\theta = 45°$ under combined in-plane loading are verified. The results are compared with those given by Jaunky et al. (1995) using Rayleigh-Ritz and FEMs and also those given by Kennedy and Prabhakara (1979).

10. Conclusion

An DQ methodology is introduced to study static and stability analyses of irregular quadrilateral straight-sided thin plates. The methodology requires less computation for the evaluation of the weighting coefficients in comparison with other developed DQ methodologies for fourth order partial differential equations which have used the first order derivatives as the second degrees of freedom. Through the

Table 7
Convergence behavior of critical buckling coefficients $K = (N_x a^2/\pi^2 D)$ of rhombic plates ($N_x \neq 0$)

Boundary type	Method	N_ξ	Skew angle			
			90°	75°	60°	45°
S–S–S–S	Present	7	4.0000	4.3970	5.9445	10.5535
		9	4.0000	4.3943	5.9057	10.1344
		13	4.0000	4.3930	5.8804	9.9440
		15	4.0000	4.3928	5.8751	9.8997
		19	4.0000	4.3926	5.8693	9.8456
	Wang et al. (1991)		4.0000	4.39	5.98	9.87
	Fried and Schmitt (1972)		4.0		5.91	10.2
	York and Williams (1995)		4.0	4.74	7.63	13.54
C–C–C–C	Present	7	9.8152	10.6312	13.9785	24.8297
		9	10.0787	10.8309	13.5050	20.5022
		13	10.0730	10.8346	13.5381	20.1088
		17	10.0740	10.8347	13.5380	20.1053
	Wang et al. (1991)		10.08	10.89	13.75	20.69
	York and Williams (1995)		10.07	10.87	13.58	20.21
S–C–S–C	Present	7	7.6420[a]	8.2501	10.6297	17.8800
		9	7.6879	8.2938	10.4882	15.9530
		15	7.6913	8.30188	10.5015	15.9156
		19	7.6913	8.30191	10.5018	15.9160
	Wang et al. (1991)		7.7	8.3	10.6	16.4
S–F–S–F	Present	7	0.95233	1.07908	1.5722	2.9885
		9	0.95231	1.07320	1.5450	2.9142
		13	0.95231	1.07000	1.5300	2.8673
		17	0.95231	1.06890	1.5236	2.8294
		19	0.95231	1.0686	1.5215	2.8146
	Wang et al. (1991)		0.95	1.1	1.5	2.8
C–F–C–F	Present	7	3.8334	4.2152	5.4137	11.142
		9	3.9209	4.2960	6.2827	8.4145
		13	3.9182	4.2838	5.6289	8.2154
		15	3.9183	4.2822	5.6265	8.1817
		19	3.9185	4.2810	5.6199	8.1452

[a] Exact solution = 7.69 (Timoshenko and Woinowsky-Krieger, 1959).

Table 8
Convergence study of critical buckling coefficient ($K = (N_x a^2)/(\pi^2 D)$) of trapezoidal plates ($N_x = N_y \neq 0$; $a/b = 1$; $\beta = 0$)

Boundary type	c/b	$N_\xi = N_\eta$				
		11	13	15	17	19
S–S–S–S	0.2	3.8233	3.8233	3.8233	3.8233	3.8233
	0.4	3.1197	3.1196	3.1196	3.1195	3.1195
	0.6	2.5976	2.5976	2.5976	2.5975	2.5975
C–C–C–C	0.2	10.710	10.703	10.703	10.703	10.703
	0.4	8.7320	8.7307	8.7308	8.7308	8.7308
	0.6	7.0952	7.0951	7.0951	7.0951	7.0951
S–C–S–C	0.2	5.1623	5.1627	5.1629	5.1630	5.1630

Table 9
Convergence study of critical buckling coefficient ($K = (N_x a^2)/(\pi^2 D)$) of unsymmetrical trapezoidal plates ($N_x = N_y \neq 0$; $a/b = 1.5$; $c/b = 0.4$)

β	$N_\xi = N_\eta$						
	9	11	13	15	17	19	21
10°	5.4824	5.4824	5.4824	5.4824	5.4824	5.4824	5.4824
20°	5.78	5.7862	5.7861	5.7860	5.7860	5.7859	5.7859

Table 10
Accuracy and convergence study of critical buckling coefficients ($K = (N_x a^2)/(\pi^2 D)$) for simply supported skew plates with skew angle $\theta = 45°$ under combined in-plane loading ($a/b = 1$)

Method	$N_\xi = N_\eta$	$N_x = N_y = 1$		$N_x = N_y = 0$
		$N_{xy} = 0$	$2N_{xy} = 1$	$N_{xy} = 0$
Present	5	4.3369	5.4237	3.6020
	7	3.9412	4.7137	3.3430
	9	3.8678	4.5970	3.2911
	11	3.8332	4.5493	3.2650
	13	3.8124	4.5212	3.2490
	15	3.7984	4.5026	3.2383
	17	3.7885	4.4894	3.2307
	19	3.7812	4.4797	3.2250
Rayleigh-Ritz (Jaunky et al., 1995)		3.938	4.697	3.342
STAGS (Jaunky et al., 1995)		3.656	4.334	3.119
Kennedy and Prabhakara (1979)		3.38	4.00	2.900

methodology, the boundary conditions are implemented accurately. The accuracy, convergence, and stability of the solution procedure results were studied through different examples for static as well as stability analysis of irregular skew plates at acute angles under different boundary conditions including the free-edge boundary type. The results were compared with those of other DQMs as well as other numerical techniques.

Appendix A

The transformation matrices may be obtained from Eq. (5) as

$$[J_{11}] = \begin{bmatrix} x_{,\xi} & y_{,\xi} \\ x_{,\eta} & y_{,\eta} \end{bmatrix}, \qquad [J_{21}] = \begin{bmatrix} x_{,\xi\xi} & y_{,\xi\xi} \\ x_{,\eta\eta} & y_{,\eta\eta} \\ x_{,\xi\eta} & y_{,\xi\eta} \end{bmatrix} \tag{A.1}$$

$$[J_{22}] = \begin{bmatrix} x_{,\xi}^2 & y_{,\xi}^2 & x_{,\xi}y_{,\xi} \\ x_{,\eta}^2 & y_{,\eta}^2 & x_{,\eta}y_{,\eta} \\ x_{,\xi}x_{,\eta} & y_{,\xi}y_{,\eta} & \frac{1}{2}(x_{,\xi}y_{,\eta} + x_{,\eta}y_{,\xi}) \end{bmatrix} \tag{A.2}$$

$$[J_{31}] = \begin{bmatrix} x_{,\xi\xi\xi} & y_{,\xi\xi\xi} \\ x_{,\eta\eta\eta} & y_{,\eta\eta\eta} \\ x_{,\xi\xi\eta} & y_{,\xi\xi\eta} \\ x_{,\xi\eta\eta} & y_{,\xi\eta\eta} \end{bmatrix} \tag{A.3}$$

$$[J_{32}] = \begin{bmatrix} 3x_{,\xi}x_{,\xi\xi} & 3y_{,\xi}y_{,\xi\xi} & 3(x_{,\xi\xi}y_{,\xi} + y_{,\xi\xi}x_{,\xi}) \\ 3x_{,\eta}x_{,\eta\eta} & 3y_{,\eta}y_{,\eta\eta} & 3(x_{,\eta\eta}y_{,\eta} + y_{,\eta\eta}x_{,\eta}) \\ x_{,\xi\xi}x_{,\eta} + 2x_{,\xi\eta}x_{,\xi} & y_{,\xi\xi}y_{,\eta} + 2y_{,\xi\eta}y_{,\xi} & x_{,\xi\xi}y_{,\eta} + y_{,\xi\xi}x_{,\eta} + 2x_{,\xi\eta}y_{,\eta} + 2y_{,\xi\eta}x_{,\xi} \\ x_{,\eta\eta}x_{,\xi} + 2x_{,\xi\eta}x_{,\eta} & y_{,\xi\xi}y_{,\xi} + 2y_{,\xi\eta}y_{,\eta} & x_{,\xi\xi}y_{,\eta} + y_{,\xi\xi}x_{,\eta} + 2x_{,\xi\eta}y_{,\eta} + 2y_{,\xi\eta}x_{,\xi} \end{bmatrix} \quad (A.4)$$

$$[J_{33}] = \begin{bmatrix} x_{,\xi}^3 & y_{,\xi}^3 & 3x_{,\xi}^2 y_{,\xi} & 3x_{,\xi}y_{,\xi}^2 \\ x_{,\eta}^3 & y_{,\eta}^3 & 3x_{,\eta}^2 y_{,\eta} & 3x_{,\eta}y_{,\eta}^2 \\ x_{,\xi}^2 x_{,\eta} & y_{,\xi}^2 y_{,\eta} & x_{,\xi}^2 y_{,\eta} + 2x_{,\xi}x_{,\eta}y_{,\xi} & y_{,\xi}^2 x_{,\eta} + 2x_{,\xi}y_{,\eta}y_{,\xi} \\ x_{,\eta}^2 x_{,\xi} & y_{,\eta}^2 y_{,\xi} & x_{,\eta}^2 y_{,\xi} + 2x_{,\xi}x_{,\eta}y_{,\eta} & y_{,\eta}^2 x_{,\xi} + 2x_{,\eta}y_{,\eta}y_{,\xi} \end{bmatrix} \quad (A.5)$$

Appendix B

The effective shear forces may be written as

$$\{Q_n\}^{\mathrm{T}} = \{n\}^{\mathrm{T}} \begin{Bmatrix} \dfrac{\partial M_x}{\partial x} + \dfrac{\partial M_{xy}}{\partial y} \\ \dfrac{\partial M_y}{\partial y} + \dfrac{\partial M_{xy}}{\partial x} \end{Bmatrix} + [\, x_s \quad y_s \,] \begin{Bmatrix} \dfrac{\partial M_{sn}}{\partial x} \\ \dfrac{\partial M_{sn}}{\partial y} \end{Bmatrix} \quad (B.1)$$

In the above equation,

$$\{n\}^{\mathrm{T}} = [\, n_x \quad n_y \,], \quad \left[\frac{\partial x}{\partial s} \quad \frac{\partial y}{\partial s} \right] = [\, -n_y \quad n_x \,]$$

where n_x, n_y are the components of the unit normal to the boundary of the physical domain. Also, one may note that (Timoshenko and Woinowsky-Krieger, 1970)

$$M_{sn} = n_x n_y(-M_x + M_y) + (x_x^2 - n_y^2)M_{xy} = \{\tilde{n}\}^{\mathrm{T}}\{M\} \quad (B.2)$$

Here,

$$\{\tilde{n}\}^{\mathrm{T}} = [\, -n_x n_y \quad n_x n_y \quad (n_x^2 - n_y^2) \,], \quad \{M\}^{\mathrm{T}} = [\, M_x \quad M_y \quad M_{xy} \,]$$

Using the constitutive law for bending moments, Eq. (B.2) may be written as

$$M_{sn} = \{\widetilde{D}\}^{\mathrm{T}}\{K\} \quad (B.3)$$

where

$$\{\widetilde{D}\}^{\mathrm{T}} = \{\tilde{n}\}^{\mathrm{T}}[D][\bar{I}], \quad [D] = \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ D_{12} & D_{22} & D_{23} \\ D_{13} & D_{23} & D_{33} \end{bmatrix}$$

Also, $\{K\}$ and $[\bar{I}]$ are given by Eqs. (27) and (21). The derivative of M_{ns} can be evaluated as

$$\frac{\partial M_{sn}}{\partial x} = \frac{\partial \{\widetilde{D}\}^{\mathrm{T}}}{\partial x}\{K\} + \{\widetilde{D}\}^{\mathrm{T}}\frac{\partial \{K\}}{\partial x} \quad (B.4)$$

$$\frac{\partial M_{sn}}{\partial y} = \frac{\partial \{\widetilde{D}\}^{\mathrm{T}}}{\partial y}\{K\} + \{\widetilde{D}\}^{\mathrm{T}}\frac{\partial \{K\}}{\partial y} \quad (B.5)$$

For a quadrilateral straight-sided plate, the first derivatives on the right hand side of Eqs. (B.4) and (B.5) are zero. Also, in order to do transformation more easily and systematic for programming, one may note that (Tham et al., 1988)

$$\frac{\partial K}{\partial x} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} \frac{\partial^3 w}{\partial x^3} \\ \frac{\partial^3 w}{\partial y^3} \\ \frac{\partial^3 w}{\partial x^2 \partial y} \\ \frac{\partial^3 w}{\partial x \partial y^2} \end{Bmatrix}, \qquad \frac{\partial K}{\partial y} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \frac{\partial^3 w}{\partial x^3} \\ \frac{\partial^3 w}{\partial y^3} \\ \frac{\partial^3 w}{\partial x^2 \partial y} \\ \frac{\partial^3 w}{\partial x \partial y^2} \end{Bmatrix} \tag{B.6}$$

Using the constitutive law for bending and rearranging terms, the first term in Eq. (B.1), i.e., Q_n, becomes,

$$Q_n = \left\{ \tilde{\tilde{D}} \right\}^{\mathrm{T}} \begin{Bmatrix} \frac{\partial^3 w}{\partial x^3} \\ \frac{\partial^3 w}{\partial y^3} \\ \frac{\partial^3 w}{\partial x^2 \partial y} \\ \frac{\partial^3 w}{\partial x \partial y^2} \end{Bmatrix} \tag{B.7}$$

where

$$\{\tilde{\tilde{D}}\} = \begin{Bmatrix} (n_x D_{11} + n_y D_{16}) \\ (n_x D_{23} + n_y D_{22}) \\ (3 n_x D_{13} + n_y (D_{12} + 2 D_{33})) \\ (n_x (D_{12} + 2 D_{33}) + 3 n_y D_{23}) \end{Bmatrix}$$

Using Eqs. (B.4)–(B.6), Eq. (B.1) becomes

$$\{F\}^{\mathrm{T}} \begin{Bmatrix} \frac{\partial^3 w}{\partial x^3} \\ \frac{\partial^3 w}{\partial y^3} \\ \frac{\partial^3 w}{\partial x^2 \partial y} \\ \frac{\partial^3 w}{\partial x \partial y^2} \end{Bmatrix} = 0 \tag{B.8}$$

where

$$\{F\}^{\mathrm{T}} = \{\tilde{\tilde{D}}\}^{\mathrm{T}} + \frac{\partial x}{\partial s} \{\tilde{D}\} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} + \frac{\partial y}{\partial s} \{\tilde{D}\} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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